Journal of Modern Optics
September 2012, 1–8

Entanglement generation in relativistic quantum fields

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(*September 2012*)

We present a general, analytic recipe to compute the entanglement that is generated between arbitrary, discrete modes of bosonic quantum fields by Bogoliubov transformations. Our setup allows the complete characterization of the quantum correlations in all Gaussian field states. Additionally, it holds for all Bogoliubov transformations. These are commonly applied in quantum optics for the description of squeezing operations, relate the mode decompositions of observers in different regions of curved spacetimes, and describe observers moving along non-stationary trajectories. We focus on a quantum optical example in a cavity quantum electrodynamics setting: an uncharged scalar field within a cavity provides a model for an optical resonator, in which entanglement is created by non-uniform acceleration. We show that the amount of generated entanglement can be magnified by initial single-mode squeezing, for which we provide an explicit formula. Applications to quantum fields in curved spacetimes, such as an expanding universe, are discussed.

Keywords: entanglement generation; Bogoliubov transformations; cavity quantum electrodynamics; squeezed light; non-uniform motion; curved spacetimes;

1. Introduction

Over the past decade the discipline of relativistic quantum information has received much attention (see Ref. [1–7] for a selection and Refs. [8, 9] for reviews). Its aim is the study of the resources and tasks of quantum information science in the context of relativity. In particular, finding suitable ways to store and process information is a main goal.

It has therefore been a central focus of previous efforts to identify the *degradation* effects due to relativistic, accelerated motion [10–16] and spacetime curvature [17, 18]. At the centre of these deteriorations lie two ingredients: the Bogoliubov transformation of the mode operators and the presence of a horizon. While the Bogoliubov transformations are responsible for the creation and shift of excitations, the horizon causes some of these excitations to be inaccessible to the observer. This usually leads to a loss of information and, consequently, a degradation of the quantum correlations. A rare exception of a situation where entanglement is *generated* by acceleration can be found in Ref. [19].

In Ref. [20] cavities were proposed as a suitable way of storing and processing information in the relativistic context. Cavities, represented by appropriate boundary conditions, can be uniformly or non-uniformly accelerated without creating a horizon. Nevertheless, the absence of an event horizon does not guarantee complete access to all modes of a quantum field. In particular, it was shown in Refs. [21, 22] how the entanglement between a non-uniformly moving cavity and an inertial reference cavity is degraded when only particular modes are being considered. In

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ISSN: 0950-0340 print/ISSN 1362-3044 online

DOI: 10.1080/09500340.2012.712725

this case the information loss can be prevented by appropriately timing the periods of uniform acceleration and inertial motion.

However, the role of the Bogoliubov transformations lies foremost in the *generation of entanglement*, an effect that can be found for a wide variety of situations. In quantum optics they are used for the description of single-mode as well as multi-mode squeezing operations, see, e.g., Ref. [23]. Entanglement is also created between modes of quantum fields in curved spacetimes [24, 25]. And, recently, entanglement generation has been demonstrated and studied quantitatively for non-uniform cavity motion in flat Minkowski spacetime in Refs. [26, 27].

Here we present a framework that generalises and significantly simplifies the previous approaches on entanglement generation in relativistic quantum information. We employ continuous variable techniques [28] that allow us to investigate the quantum correlations that are created by arbitrary Bogoliubov transformations of any discrete number of modes. The transformed state of these modes is described by its covariance matrix, which completely determines the entanglement of the system. Results for any given Bogoliubov transformation can thus be obtained analytically in principle, if the transformation coefficient are known.

We apply our method to a quantum optical setting: an optical cavity that is moving non-uniformly is modelled by confining an uncharged scalar field by appropriate boundary conditions [21]. The cavity is following a trajectory that consists of segments of uniform acceleration and inertial motion. We work in a perturbative regime where the products of the cavity's width and the individual accelerations in all segments are small, but all calculations can be carried out analytically. The Bogoliubov coefficients can be expressed as a Maclaurin series in h, a parameter which represents the product of the acceleration at the centre of the cavity and the cavity's width δ. The creation of entanglement for initially uncorrelated, single-mode squeezed states of the field is quantified in terms of the expansion coefficients of the power series. We find that the entanglement generation in this fully relativistic setting can be enhanced by initial single-mode squeezing. This provides an additional controllable parameter for possible experimental setups to enhance the visibility of the effect.

We further emphasize the applicability of our approach beyond cavity quantum electrodynamics by verifying the results obtained for an expanding universe in Ref. [24] as a curved spacetime examples.

This article is structured as follows. In Sec. 2 we establish the basic description of Bogoliubov transformations for arbitrary Gaussian initial states in terms of the corresponding symplectic transformation of the covariance matrix. In Sec. 3 we revise the description of non-uniformly moving cavities from Refs. [21, 22] and, subsequently, apply the formalism of Sec. 2 to study the entanglement creation by motion in this scenario.

2. Bogoliubov transformation of Gaussian states

It is the aim of this section to present a general machinery that allows a simple characterisation of the mode entanglement of bosonic quantum fields. In particular we aim to construct a framework that works for any specified Bogoliubov transformation.

To this end, let us consider an arbitrary discrete set of mode functions $\{\phi_n \,|\, n = 1, 2, 3, \dots\}$ of a bosonic quantum field, let us assume a scalar field for simplicity, with associated annihilation and creation operators, a_n and $a_n^\dagger$, respectively. The field operators satisfy the canonical commutation relations $[a_m, a_n] = [a_m^\dagger, a_n^\dagger] = 0$ and $[a_m, a_n^\dagger] = \delta_{mn}$. The functions ϕ_n, which we take to be the solutions to a (relativistic) field equation, e.g., the Klein-Gordon equation, form a complete set of orthonormal modes with respect to a chosen scalar product, i.e., $(\phi_m, \phi_n) = \delta_{mn}$, see, e.g., Ref. [29]. In the usual Fock representation the vacuum state is annihilated by all a_n, i.e., $a_n \,|\, 0 \,\rangle = 0$, $\forall\, n$, while particle states are created by the (repeated) action of the creation operators $a_n^\dagger$.

We can then perform a *Bogoliubov transformation* that relates our initial modes ϕ_n

and the associated operators a_n and $a_n^\dagger$ to another (complete, orthonormal) set of modes $\{\tilde{\phi}_n \,|\, n = 1, 2, 3, \ldots\}$, with operators $\tilde{a}_n$ and $\tilde{a}_n^\dagger$, respectively, i.e.,

$$\tilde{a}_m \,=\, \sum_n \left(\, \alpha_{mn}^* \, a_n \,-\, \beta_{mn}^* \, a_n^\dagger \,\right). \tag{1}$$

Here we have used the conventions of Ref. [29]. The asterisk denotes complex conjugation and $\alpha_{mn} = (\tilde{\phi}_m, \phi_n)$ and $\beta_{mn} = -(\tilde{\phi}_n, \phi_m^*)$ are the coefficients of the Bogoliubov transformation.

This transformation can represent a physical operation in a laboratory, e.g., a squeezing operation in an optical setup, or a change of potential in a harmonic oscillator chain of trapped atoms. Alternatively, the transformation may describe a change of observer in a (curved) spacetime, e.g., a black hole spacetime or a cosmological model, or relate the modes at different times of a single observer that is moving non-uniformly. For a continuous spectrum of solutions the sum in Eq. (1) has to be replaced by an appropriate integral, but we will specialise to discrete sets of modes in the following.

Analogously to before, the vacuum state $|\,\tilde{0}\,\rangle$ of the new mode decomposition is defined by the property $\tilde{a}_n \,|\,\tilde{0}\,\rangle = 0$, $\forall\, n$. However, an important feature of the Bogoliubov transformations is the non-equivalence of the corresponding vacua, i.e., $a_m \,|\,\tilde{0}\,\rangle = 0$ only if $\beta_{mn} = 0$, $\forall\, n$. Consequently, the vacua generally have a nontrivial relationship, see, e.g., Ref. [30].

To study the quantum correlations between particular modes two more steps are required. The complete Fock state basis of the original modes has to be expressed in terms of the Fock states of the transformed modes. Subsequently, the subset of modes that is not considered must be traced over. While the first step can generally be accomplished, the tracing procedures are usually cumbersome.

In the following we demonstrate how these computations can be simplified significantly. Let us switch to the covariance matrix formalism discussed in Ref. [28]. This means, instead of the complete Fock space description of the quantum state, we consider only the matrix σ with components

$$\sigma_{ij} \,=\, \langle\, \mathrm{X}_i \mathrm{X}_j \,+\, \mathrm{X}_j \mathrm{X}_i \,\rangle \,-\, 2\, \langle\, \mathrm{X}_i \,\rangle \langle\, \mathrm{X}_j \,\rangle\ . \tag{2}$$

The quadrature operators X_i are chosen to be the generalised positions and momenta, i.e., $\mathrm{X}_{(2n-1)} = \frac{1}{\sqrt{2}}(a_n + a_n^\dagger)$ and $\mathrm{X}_{(2n)} = \frac{-i}{\sqrt{2}}(a_n - a_n^\dagger)$, where the index $n = 1, 2, 3, \ldots$ labels the modes, and $\langle\, \mathcal{O}\,\rangle$ denotes the expectation value of the operator $\mathcal{O}$. The real, symmetric covariance matrix σ together with the vector of first moments X_i completely characterises all Gaussian states, i.e., states that can be described by quasi-probability distributions of Gaussian shape in phase space, see, e.g., Ref. [28]. However, the covariance matrix itself is a sufficient description of all properties pertaining to entanglement. While these continuous variable tools have been frequently employed in quantum optics settings, their application to quantum field theory and relativistic quantum information was previously explored only in limited situations [12, 17].

Unitary transformations in the Fock space are represented by symplectic transformations in phase space. A transformation S is called symplectic, if it leaves the symplectic form Ω invariant, i.e., $S\,\Omega\,S^T = \Omega$, where $[X_i, X_j] = i\Omega_{ij}$ and $(S_{ij})^T = (S_{ji})$. From Eq. (1) it is straightforward to find the expression for the symplectic representation S of a Bogoliubov transformation in terms of its general coefficients α_{mn} and β_{mn}. The matrix S is decomposed into 2×2 blocks $\mathcal{M}_{mn}$ as

$$S \,=\, \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} & \ldots \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} & \ldots \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3}$$

where the sub-blocks are given by

$$\mathcal{M}_{mn} = \begin{pmatrix} \Re(\alpha_{mn} - \beta_{mn}) & \Im(\alpha_{mn} + \beta_{mn}) \\ -\Im(\alpha_{mn} - \beta_{mn}) & \Re(\alpha_{mn} + \beta_{mn}) \end{pmatrix} . \tag{4}$$

Here $\Re(z)$ and $\Im(z)$ denote the real part and imaginary part of the complex number z respectively. The transformed covariance matrix $\tilde{\sigma}$ is then simply obtained as

$$\tilde{\sigma} = S \, \sigma \, S^T . \tag{5}$$

The crucial technical simplification lies in the ensuing step. The partial trace over any subset of modes $\mathcal{E} \subset \{n = 1, 2, 3, \ldots\}$ can be computed trivially by eliminating all rows and columns corresponding to modes $m \in \mathcal{E}$. This technique applies for any initial state σ. However, if we restrict ourselves to Gaussian states, the unitarity of the transformation ensures that $\tilde{\sigma}$ remains Gaussian, which allows us to use the specifically useful tools for the quantification of Gaussian entanglement [31]. In particular, we are interested in the generation of entanglement from initially uncorrelated states. In this case the covariance matrix σ has a block diagonal structure, $\sigma = \operatorname{diag}\{\psi_n\}$, where the 2×2 blocks ψ_n are locally equivalent to single-mode squeezed states with squeezing parameters s_n, i.e.,

$$\psi_n = \begin{pmatrix} e^{s_n} & 0 \\ 0 & e^{-s_n} \end{pmatrix} . \tag{6}$$

Let us now consider the transformed state of two modes, labelled k and k' respectively. In other words, $\mathcal{E} = \{n = 1, 2, 3, \ldots | n \neq k, k'\}$ is the set of all modes except the chosen pair k and k'. The transformed covariance matrix $\tilde{\sigma}_{kk'}$ of these two modes is then given by

$$\tilde{\sigma}_{kk'} = \begin{pmatrix} C_{kk} & C_{kk'} \\ C_{k'k} & C_{k'k'} \end{pmatrix} , \tag{7}$$

where $C_{ij} = \sum_n \mathcal{M}_{in} \, \psi_n \, \mathcal{M}^T_{jn}$.

Let us now turn to the quantification of the entanglement that is generated between the modes k and k'. The quantity that encodes all the information about the entanglement of symmetric two-mode Gaussian states is the smallest symplectic eigenvalue $\widehat{\nu}_-$ of $\widehat{\sigma}_{kk'} = T_{k'} \, \tilde{\sigma}_{kk'} \, T_{k'}$, where $T_{k'} = \operatorname{diag}\{1, 1, 1, -1\}$ represents the partial transposition of mode k', see Ref. [28]. For this class of states the Gaussian measures of entanglement, such as the (logarithmic) negativity or the Gaussian entanglement of formation, are monotonously decreasing functions of $\widehat{\nu}_-$. However, the two-mode Gaussian state $\tilde{\sigma}_{kk'}$ of Eq. (7) is generally not symmetric with respect to the interchange of k and k'. Consequently, different entanglement measures can exhibit different ordering of two entangled states.

With this in mind we specialise to the use of the negativity $\mathcal{N}$ as an entanglement measure. It has the advantage of being easily computable, i.e., the negativity of $\sigma_{kk'}$ is given in terms of the smallest symplectic eigenvalue $\widehat{\nu}_-$ of $\widehat{\sigma}_{kk'}$ by the simple formula

$$\mathcal{N} = \max\{0, (1 - \widehat{\nu}_-)/2\widehat{\nu}_-\} . \tag{8}$$

Furthermore, it can be easily compared to results obtained for non-Gaussian states in Ref. [26]. The smallest symplectic eigenvalue is obtained by diagonalising the matrix $\widehat{\sigma}_{kk'}$ by a symplectic operation D, such that $D \, \widehat{\sigma}_{kk'} \, D^T = \operatorname{diag}\{-\widehat{\nu}_-, \widehat{\nu}_-, -\widehat{\nu}_+, \widehat{\nu}_+\}$, where $0 \leq \widehat{\nu}_- \leq \widehat{\nu}_+$. The quantities $\pm\widehat{\nu}_\pm$ can be computed as the eigenvalues of $i\Omega\widehat{\sigma}_{kk'}$ in a straightforward way. For $0 \leq \widehat{\nu}_- < 1$

the state $\tilde{\sigma}_{kk'}$ is entangled.

Other well known entanglement measures for Gaussian states, e.g., Gaussian entanglement of formation, can also be calculated straightforwardly for two-mode states in principle [28]. The transformed state will generally be a mixed state but all calculations can be done analytically, provided that the Bogoliubov coefficients are given and the infinite sums in Eq. (7) are convergent. In Sec. 3 we discuss an example of a discrete spectrum for which the Bogoliubov coefficients are given as a perturbative expansion and the sums converge [21, 22]. We compute the entanglement generated from non-uniform cavity motion for particular initial states of interest. For continuous spectra, however, the corresponding integrals are often known to be divergent, see, e.g., Ref. [32], thus rendering any conclusive statements about the entanglement generated between the modes fruitless, unless the coefficients have a simple structure.

Such a situation presents itself in the case of a charged, scalar field in an expanding universe, which was discussed in Ref. [24]. There the spectrum of the quantum field is continuous, but the Bogoliubov transformation between the asymptotically flat remote past and future couple only modes of opposite momenta. In this fashion, the transformation provides an effective discretization and we can reproduce the results of Ref. [24] with our methods.

3. Non-uniform cavity motion

An example for a discrete, bosonic spectrum is obtained by confining a scalar quantum field to a cavity by appropriate Dirichlet boundary conditions in $(1+1)$ dimensions. As proposed in Ref. [21] the rigid cavity can follow a worldline that is composed of segments of inertial motion and uniform acceleration. This non-uniform motion generates non-trivial Bogoliubov coefficients, which result in a generation of entanglement between the modes inside one cavity [26] and, consequently, lead to a degradation of initial entanglement between modes in different cavities [21].

For a quantitative description the Bogoliubov transformations are expanded as a Maclaurin series in the small, dimensionless parameter h, i.e.,

$$\alpha = \alpha^{(0)} + \alpha^{(1)} + \alpha^{(2)} + O(h^3)\,, \tag{9a}$$

$$\beta = \beta^{(1)} + \beta^{(2)} + O(h^3)\,, \tag{9b}$$

where the superscripts $^{(n)}$ denote quantities proportional to h^n. The parameter h is the product of the cavity's length in its instantaneous rest frame and the proper acceleration at the centre of the cavity. Here we use units such that Planck's constant and the speed of light are dimensionless constants, $\hbar = c = 1$, and the entity $O(x)/x$ is bounded as x goes to 0. Furthermore, the coefficient $\alpha^{(0)}$ must include the phases of the free time evolution in the uniformly accelerated and inertial segments, while it reduces to the identity for vanishing accelerations. We therefore have $\alpha^{(0)}_{mn} = \delta_{mn} G_m$, where G_m is a mode-dependent phase factor of unit magnitude, i.e., $|G_m| = 1$. Additionally, we find that the linear corrections vanish on the diagonal, i.e., $\alpha^{(1)}_{nn} = \beta^{(1)}_{nn} = 0$.

The perturbative calculations require a closer inspection of the techniques for the calculation of the symplectic eigenvalues $\widehat{\nu}_-$ from Sec. 2. Let us assume that the initial state $\sigma_{kk'}$ is transformed by the Bogoliubov transformation according to

$$\tilde{\sigma}_{kk'} = \sigma_{kk'} + \sigma^{c}_{kk'}\,, \tag{10}$$

where$\sigma^{c}_{kk'}$ is a small correction to the initial state. For uncorrelated, pure initial states $\sigma_{kk'}$, which we want to study here, the unperturbed symplectic eigenvalues $\widehat{\nu}^{(0)}_{\pm}$ of $T_{k'}\,\sigma_{kk'}\,T_{k'}$ are

degenerate, i.e., $\widehat{\nu}_+^{(0)} = \widehat{\nu}_-^{(0)} = 1$. This requires the diagonalisation of the subspaces of the degenerate eigenvalues of the perturbation $\sigma^c_{kk'}$ to obtain the corrected smallest symplectic eigenvalue $\widehat{\nu}_- = 1 \pm \widehat{\nu}_-^c$. In other words, the corrections $\pm\widehat{\nu}_-^c$ to $\widehat{\nu}_\pm^{(0)} = 1$ are given by the eigenvalues of the matrix (γ^c_{ij}) with components

$$\gamma^c_{ij} = \langle\, e_{i\pm} \,|\, i\,\Omega\, \widehat{\sigma}^c_{kk'} \,|\, e_{j\pm} \,\rangle\;, \quad (i,j=1,2)\,, \tag{11}$$

where $|\, e_{j\pm}\,\rangle$ is the j-th eigenvector of $i\,\Omega\,\widehat{\sigma}_{kk'}$ corresponding to eigenvalue ±1. An important feature of the unperturbed state $\sigma_{kk'}$, and consequently of the eigenvectors of $i\,\Omega\,\widehat{\sigma}_{kk'}$, is its time dependence due to the free time evolution of the modes, i.e., $\sigma_{kk'} \rightarrow R\,\sigma_{kk'}R^T$. The local rotation R is represented by an orthogonal, block diagonal matrix $R = \mathrm{diag}\{R_n\}$, where the block of the n-th mode can be written as $R_n = \Re(G_n)\mathbb{1}_2 + i\,\Im(G_n)\sigma_y$, and σ_y is the usual Pauli matrix. The leading order correction to the negativity (8) is given by $|\widehat{\nu}_-^c|/2$. As a particular example we can study the case of symmetric, initial single-mode squeezing in the modes k and k', with (real) squeezing parameters $s_k = s_{k'} = s$ (see Eq. (6)). In this situation the leading order corrections to the negativity become

$$\mathcal{N} = \Big(\Re(G_k^*\beta^{(1)}_{kk'})^2 + (\,\Im(G_k^*\beta^{(1)}_{kk'})\cosh(s) - \Im(G_k^*\alpha^{(1)}_{kk'})\sinh(s))^2\Big)^{1/2} + O(h^2)\,. \tag{12}$$

For modes of opposite parity, i.e., if $(k+k')$ is odd, the linear coefficients are non-zero see Ref. [21]. In the case of vanishing squeezing parameters, $s_k = s_{k'} = 0$, the expression in (12) reduces to $\mathcal{N} = |\beta^{(1)}_{kk'}|$, which is consistent with the expression for the entanglement that is generated from the bosonic vacuum in the Fock representation, see Ref. [26]. As can easily be seen from Eq. (12) the initial single mode squeezing introduces a non-negative term into the negativity. Thus the generated negativity is always enhanced with respect to the vacuum case. An example for the effect of single-mode squeezing on the generation of entanglement is shown in Fig. 1(a) for a particular travel scenario of the cavity. It can be readily observed, that the negativity grows with e^s for $e^s \gg e^{-s}$. While this fact can be utilised to enhance the visibility of the entanglement generation, it also limits the validity of the perturbative regime, which requires further investigation.

A transparent method to quantify the perturbation of the initial state is the analysis of the system's mixedness. As discussed in Ref. [21, 27] the Bogoliubov coefficients that relate the modes k and k' can be consistently renormalized to represent a unitary transformation on the subspace of these two modes. This *two-mode truncation*, which leaves the linear coefficients $\alpha^{(1)}_{kk'}$ and $\beta^{(1)}_{kk'}$ invariant, can be implemented if k and k' have opposite parity. The state of a truncated system of this type will undergo a unitary evolution due to the cavity motion. In particular, the determinants of the individual mode subspaces, $\det C_{kk}$ and $\det C_{k'k'}$, will be identical to leading order in h, i.e., the state is symmetric.

We then exploit the fact that any pure, symmetric two-mode Gaussian state is locally equivalent to a two-mode squeezed state [28]. This enables us to express the corresponding two-mode squeezing parameter r as a Maclaurin series in h by using $|r| = \frac{1}{2}\,\mathrm{arsinh}\sqrt{-\det C_{kk'}}$. We recover the expression of Eq. (12) for the negativity, i.e., $\mathcal{N} = |r|$ as is expected for two-mode squeezed states. This suggests that Eq. (12) is valid as long as the perturbation to the mixedness of the system is small, i.e., as long as $|\det\tilde{\sigma}_{kk'} - 1| \ll 1$. We compute the determinant of the transformed state for the symmetrically single-mode squeezed initial state to be

$$\begin{aligned}\det\tilde{\sigma}_{kk'} = 1 \,+\, 4(f^{\beta}_{k\neg k'} + f^{\beta}_{k'\neg k})(\cosh s + 1) \,+\, 4(f^{\alpha}_{k\neg k'} + f^{\alpha}_{k'\neg k})(\cosh s - 1)\\ -\, 4\sinh s \sum_{n\neq k,k'} \Re\big(\,\alpha^{(1)}_{nk}\beta^{(1)*}_{nk} \,+\, \alpha^{(1)}_{nk'}\beta^{(1)*}_{nk'}\,\big)\,,\end{aligned} \tag{13}$$

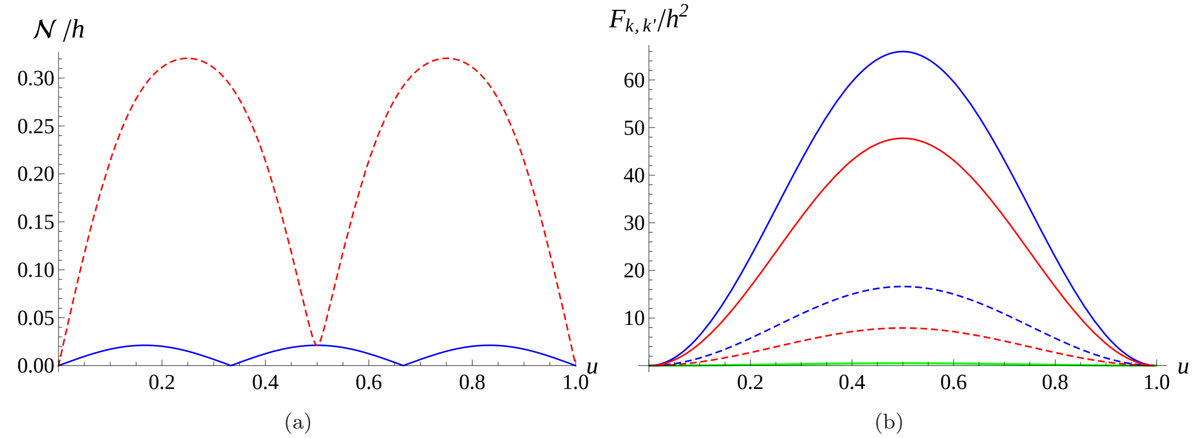

Figure 1. The leading order contribution $\mathcal{N}/h$ to the negativity, see Eq. (12), is shown in Fig. 1(a) for a massless scalar field confined to a moving cavity of width δ. The cavity's trajectory is inertial outside a single segment of uniform, linear acceleration h/δ of duration τ, as experienced at the centre of the cavity. The Bogoliubov coefficients for this travel scenario can be found in Ref. [21]. Plots are shown for an initial state of the chosen modes $(k, k') = (1, 2)$ that is symmetrically single-mode squeezed with $s = 1$ (dashed, red) or in the (displaced) vacuum, $s = 0$ (solid, blue). Fig. 1(b) illustrates the behaviour of the quantities $F_{k,k'}/h^2 = (f^{\alpha}_{k\neg k'} + f^{\alpha}_{k'\neg k})/h^2$ for increasing mode numbers. The same field and travel scenario as in Fig. 1(a) are used, but the mode numbers are varied. From bottom to top the curves show $(k, k') = (1, 2)$ (solid, green), $(k, k') = (10, 11)$ (dashed, red), $(k, k') = (1, 10)$ (dashed, blue), $(k, k') = (20, 21)$ (solid, red) and $(k, k') = (1, 20)$ (solid, blue). All curves are plotted as functions of the temporal parameter $u := h\tau/[4\delta\,\text{atanh}(h/2)]$.

where $f^{\alpha}_{k\neg k'} = \frac{1}{2}\sum_{n\neq k'} |\alpha^{(1)}_{nk}|^2$ and $f^{\beta}_{k\neg k'} = \frac{1}{2}\sum_{n\neq k'} |\beta^{(1)}_{nk}|^2$. The α-coefficients grow with the mode numbers of the selected modes [21], while $|\beta^{(1)}_{mn}| \to 0$ if $m \to \infty$ or $n \to \infty$. We can thus constrain the range of validity of the perturbative result of Eq. (12) by the simple requirement $F_{k,k'}(\cosh s - 1) \ll 1$, where $F_{k,k'} = (f^{\alpha}_{k\neg k'} + f^{\alpha}_{k'\neg k})$. Sample plots for $F_{k,k'}/h$ are shown in Fig. 1(b). Additionally, when a massive, bosonic field is considered the mass of the field excitations must be appropriately restricted, see Ref. [21]. However, as long as these conditions are satisfied, the single-mode squeezing parameter, the choice of modes and the particular travel scenario for the cavity can be tuned to enhance the entanglement generation effect.

4. Conclusions

We have presented a general framework that allows the quantification of the entanglement that is generated between arbitrary modes of bosonic quantum fields by Bogoliubov transformations. Our setup combines techniques from quantum optics, quantum field theory and quantum information procedures to describe the quantum correlations that arise from non-uniform motion, spacetime curvature and quantum optical operations in a covariance matrix formalism. This removes the necessity for cumbersome partial tracing procedures to quantify the entanglement between field modes. For Gaussian initial states we can fully characterise the entanglement that is produced in these situations.

We have discussed a particular example, a cavity containing a relativistic quantum field, in which non-uniform motion was recently found to create entanglement [26]. The Bogoliubov coefficients for this scenario are given as a perturbative expansion and we compute the negativity, the entanglement measure of our choice, to leading order in the expansion parameter. We discuss in detail the regimes in which the perturbative calculations can be trusted.

To leading order in the expansion parameter, the transformed state of two modes of opposite parity is equivalent to a pure, two-mode squeezed state. The corresponding squeezing parameter can be used as an alternative route to quantify the entanglement of the system. We give clear criteria for the validity of this two-mode truncation of the Bogoliubov transformation. The

calculation of the mixedness of the two-mode state without the truncation provides a simple benchmark against which to judge the limits of the small parameter expansion.

By application of the methods presented here we are able to extend the previously found results to the important class of Gaussian states. We find that initial single-mode squeezing can enhance the correlations that can be generated from the vacuum, thus greatly improving the prospects of experimental verification of this effect and of its implementation in new quantum technologies.

Under some circumstances, the approach of this article even permits application to particular cases of continuous spectra of quantum fields, such as a charged, scalar field in an expanding universe [24].

Acknowledgements

We thank Gerardo Adesso, David Edward Bruschi, Andrzej Dragan, Lucia Hackermüller, Antony R. Lee and Jorma Louko for helpful discussions and comments. I.F. and N.F. acknowledge support from EPSRC [CAF Grant No. EP/G00496X/2 to I.F.]. N.F. thanks the χ-QEN collaboration for support.

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